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Values of Zeta and L-functions

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1. Introduction

In this paper, we will discuss recent investigations into the values of zeta and L-functions at $s = 0, -1, -2, \cdots$, with particular reference to $s = 0$. This is especially appropriate here since three important mathematical objects bearing Dedekind's name arise in an interesting fashion: the Dedekind zeta function, Dedekind domains, and the Dedekind eta function.

The Dedekind zeta function has been of the utmost importance in number theory since its inception. For a number field K, it is given by

$$\zeta_K(s) = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s},$$

the sum being over all (non-zero) ideals $\mathfrak{a}$ of the ring of integers $\mathfrak{O}_K$ of K. The series converges for $\mathrm{Re}(s) > 1$, and in Dedekind's time, lattice point estimates had even allowed a proof that $\zeta_K(s)$ has a first order pole at $s = 1$ with residue

$$(1) \qquad \mathrm{res}_{s=1}\, \zeta_K(s) = \frac{2^{r_1+r_2}\, \pi^{r_2}\, h(K)\, R(K)}{W(K)\sqrt{|D(K)|}}$$

where as always, K is of degree $n(K)$, has $r_1(K)$ real conjugates, $2r_2(K)$ complex conjugates, and $h(K)$, $R(K)$, $W(K)$ and $D(K)$ denote the class-number, regulator, number of roots of unity and discriminant of K respectively.

The functional equation of $\zeta_K(s)$ in its symmetric form is

$$\xi_K(s) = \xi_K(1-s)$$

where

$$\xi_K(s) = \left(\frac{|D(K)|}{2^{2r_2}\, \pi^n}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2}\, \zeta_K(s).$$

In particular, the residue at $s = 1$ provides the value,

$$(2) \qquad \lim_{s\to 0} \frac{\zeta_K(s)}{s^r} = -\frac{h(K)\, R(K)}{W(K)}$$

where $r = r_1 + r_2 - 1$ is the rank of the unit group of K.

Until recently, the residue at $s = 1$ held the limelight and the equivalent (2) was regarded as a nice curiousity. In Dedekind's time, (2) was not even available, since the

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analytic continuation to $s=0$ was not proved for general K until Hecke proved it and the functional equation in 1917. However, as we will see, the values of zeta and L-functions at zero and negative integers are nicer than the equivalent values at positive integers, are frequently special cases of more general situations, and are often more easily derived.

As an example of a more general situation, let $\mathfrak{f}$ denote an integral ideal of K and consider the Dedekind domain

$$A = \mathfrak{O}_K[\mathfrak{f}^{-1}]$$

which consists of the elements of K which are integral outside $\mathfrak{f}$. The zeta function of A is just

$$\zeta_A(s) = \zeta_K(s,\mathfrak{f}) = \zeta_K(s)\prod_{\mathfrak{p}|\mathfrak{f}}(1-\mathbb{N}(\mathfrak{p})^{-s}),$$

the zeta function of K with all the $\mathfrak{p}$ factors with $\mathfrak{p}\,|\,\mathfrak{f}$ eliminated. The rank $r(A)$ of the unit group of A equals $r(K)$ plus the number of distinct prime factors of $\mathfrak{f}$. Equation (2) holds essentially unchanged,

$$(3) \qquad \lim_{s\to 0} \frac{\zeta_A(s)}{s^{r(A)}} = -\frac{h(A)\,R(A)}{W(A)}$$

where $h(A)$ and $R(A)$ are the class-number and regulator of A. Nothing equally nice happens at $s=1$.

Zeta functions of number fields are products of Artin L-functions. The prototype for this is the case of a quadratic field K where Dirichlet demonstrated that

$$\zeta_K(s) = \zeta(s)\, L(s,\chi)$$

where χ is the corresponding Dirichlet character (mod $D(K)$) and

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}.$$

If K is complex quadratic, we get from (1) and (2) the equivalent values

$$L(1,\chi) = \frac{2\pi\, h(K)}{W(K)\sqrt{|D(K)|}},$$

$$(4) \qquad L(0,\chi) = \frac{h(K)}{W(K)/2}.$$

For example, if $K = \mathbb{Q}(\sqrt{-1})$, $W(K) = 4$, $D(K) = -4$ and

$$L(s,\chi) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + 9^{-s} - 11^{-s} + \dots .$$

It took Leibnitz to sum the series

$$L(1,\chi) = \frac{\pi}{4}$$

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whereas, "clearly"

$$L(0,\chi) = 1-1+1-1+\ldots = 1/2.$$

In fact, for any quadratic field K,

$$\sum_{n=1}^{|D(K)|} \chi(n) = 0 \tag{5}$$

and thus the sequence of partial sums of the series for $L(0,\chi)$ is periodic. Hence the series is Cesaro summable at $s = 0$ and we get

$$\begin{aligned} L(0,\chi) &= \frac{1}{|D(K)|} \sum_{N=1}^{|D(K)|} \sum_{n=1}^{N} \chi(n) \\ &= \frac{1}{|D(K)|} \sum_{n=1}^{|D(K)|} \chi(n)\,(|D(K)|+1-n). \end{aligned}$$

By (5),

$$L(0,\chi) = \sum_{n=1}^{|D(K)|} \chi(n) \left(c - \frac{n}{|D(K)|}\right)$$

holds for any c. We will soon see that $c=1/2$ is the nicest value. For complex quadratic fields, a reference to (4) shows that we have derived Dirichlet's class-number formula. This is a much easier task at $s=0$ than at $s=1$ (in particular, the Gaussian sums do not enter into the evaluation at $s=0$).

For a real quadratic field K, we have the equivalent values,

$$L(1,\chi) = \frac{2\,h(K) \log \varepsilon_K}{\sqrt{D(K)}},$$

$$L'(0,\chi) = h(K) \log \varepsilon_K$$

where ε_K is the fundamental unit of K. As an example, for $\mathbb{Q}(\sqrt{5})$ we have

$$L(s,\chi) = 1^{-s} - 2^{-s} - 3^{-s} + 4^{-s} + 6^{-s} - 7^{-s} - 8^{-s} + 9^{-s} + \ldots\,.$$

Although Dirichlet did it, it is difficult to evaluate this series at $s=1$. However, the series for $L'(0,\chi)$ is again Cesaro summable and we get

$$L'(0,\chi) = \log\left(\prod_{n=0}^{\infty} \frac{(5n+2)\,(5n+3)}{(5n+1)\,(5n+4)}\right)$$

A few terms of the product give $L'(0,\chi) \approx \log(1.6)$, or, if you know how to accelerate the convergence of the series, $L'(0,\chi) \approx \log(1.618033989)$. But in fact, we can evaluate the product directly. Indeed (C is Euler's constant),

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$$\prod_{n=0}^{\infty} \frac{(5n+2)(5n+3)}{(5n+1)(5n+4)} = \frac{(2/5)e^{2C/5}\,(3/5)e^{3C/5}}{(1/5)e^{C/5}\,(4/5)e^{4C/5}} \cdot$$

$$\cdot \prod_{n=1}^{\infty} \frac{\left(1+\frac{2/5}{n}\right)\exp\left(\frac{-2/5}{n}\right)\left(1+\frac{3/5}{n}\right)\exp\left(\frac{-3/5}{n}\right)}{\left(1+\frac{1/5}{n}\right)\exp\left(\frac{-1/5}{n}\right)\left(1+\frac{4/5}{n}\right)\exp\left(\frac{-4/5}{n}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)}$$

$$= \frac{\sin\left(\frac{2\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)}$$

The same thing works for any real quadratic field:

$$L'(0,\chi) = \log \left[\prod_{\substack{a=1 \\ (a,\, D(K))=1}}^{D(K)/2} \sin\left(\frac{a\pi}{(D(K))}\right)^{-\chi(a)} \right]$$

and this gives Dirichlet's class-number formula for real quadratic fields.

2. Dirichlet L-series at zero and negative integers

It will be convenient to discuss the values of $L(0,\chi)$ simultaneously for all χ defined modulo an integer $f>1$. For this purpose, we have to distinguish between primitive and imprimitive L-series. By $L(s,\chi)$, we will mean the L-series associated to the primitive version of χ. We will write

$$L(s,\chi,f) = L(s,\chi) \prod_{p|f} (1-\chi(p)\,p^{-s})$$

to denote the L-series corresponding to χ considered as a (possibly imprimitive) character (mod f). It is the primitive L-series with the p-factors removed where $p|f$. For example, if χ_1 is the trivial character,

$$L(s,\chi_1,f) = \zeta(s,f) = \zeta(s) \prod_{p|f} (1-p^{-s}).$$

For any character χ (mod f), we have

$$L(s,\chi,f) = \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi(a) \sum_{n=0}^{\infty} (nf+a)^{-s}$$

$$= f^{-s} \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi(a)\, \zeta(s, a/f)$$

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where

$$\zeta(s,x) = \sum_{n=0}^{\infty} (n+x)^{-s}$$

is the Hurwitz zeta function. The series for $\zeta(s,x)$ converges for $\mathrm{Re}(s)>1$ and real $x>0$ but the function possesses an analytic continuation in both s and x, multivalued in the case of x if s is not integral, and is holomorphic in s except at $s=1$ where there is a first order pole of residue one. The existence of this analytic continuation justifies the following manipulations in which we evaluate $\zeta(-k,x)$ and hence $L(-k,\chi,f)$ for $k=0,-1,-2,\cdots$.

We see that

$$(6) \qquad \frac{\partial\zeta(s,x)}{\partial x} = -s\,\zeta(s+1,x).$$

Thus, for example,

$$\frac{\partial\zeta(0,x)}{\partial x} = -\mathrm{res}_{s=1}\,\zeta(s,x) = -1$$

and so $\zeta(0,x) = -x+c$ for some c. To find c, we note that for $\mathrm{Re}(s)<1$,

$$(7) \qquad \int_0^1 \zeta(s,x)\,dx = 0$$

which we get by integrating the series term by term. Of course, the series converges only for $\mathrm{Re}(s)>1$ and then the integral is divergent. The justification makes use of the fact that $\zeta(s,x) = x^{-s} + \zeta(s,x+1)$ and the fact that $\zeta(s,x+1)$ can be integrated term by term for $\mathrm{Re}(s)>1$.

We therefore find that

$$\zeta(0,x) = -x+1/2.$$

One more application of (6) and (7) gives

$$\zeta(-1,x) = -\frac{x^2}{2}+\frac{x}{2}-\frac{1}{12}.$$

If we continue in this manner, we recursively find that $\zeta(-k,x)$ is a $(k+1)^{st}$ degree polynomial and in fact the recursions are those satisfied by the Bernoulli polynomials,

$$\zeta(-k,x) = -\frac{B_{k+1}(x)}{k+1}.$$

Many of the standard properties of Bernoulli polynomials follow from this relation. For example,

$$\sum_{n=0}^{N} n^k = \zeta(-k,0) - \zeta(-k,N+1) = \frac{B_{k+1}(N+1) - B_{k+1}(0)}{k+1}.$$

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The special case of $N = 0$ gives $B_1(1) = B_1(0) + 1$ while for $k > 0$

$$B_{k+1}(1) = B_{k+1}(0).$$

As another example, if k is odd, we have for any positive integer N,

$$\zeta(-k, 1-N) = \sum_{n=1-N}^{N-1} n^k + \zeta(-k, N) = \zeta(-k, N),$$

from which we see that for k odd

$$B_{k+1}(1-x) = B_{k+1}(x).$$

Differentiating shows that in general,

$$B_{k+1}(1-x) = (-1)^{k+1} B_{k+1}(x).$$

As a last example, the identity

$$\zeta(s, 1/2) = (2^s - 1)\, \zeta(s)$$

gives at $s = -k$, the relation

$$B_{k+1}(1/2) = (2^{-k} - 1)\, B_{k+1}(1) = (2^{-k} - 1)\, B_{k+1}(0)$$

valid for $k = 0$ and $k = -1$ as well as for $k > 0$. It is interesting to note that many of the above formulas occur in Chapter 7 of Ramanujan's second notebook [1] in the context of the relationship between $\zeta(s,x)$ and $B_{k+1}(x)$. Chapter 8 [2] is also relevant to this subject and I want to thank Bruce Berndt for bringing these references to my attention.

For L-functions, we now have at $k = 0, -1, -2, \cdots$,

$$L(-k, \chi, f) = -\frac{f^k}{k+1} \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi(a)\, B_{k+1}(a/f).$$

In particular,

$$L(0, \chi, f) = \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi(a) \left(\frac{1}{2} - \frac{a}{f}\right).$$

If $\chi(-1) = 1$ and $f > 1$, we have

$$\begin{aligned} L(0, \chi, f) &= \frac{1}{2} \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \left[\chi(a) \left(\frac{1}{2} - \frac{a}{f}\right) + \chi(f-a) \left(\frac{1}{2} - \frac{f-a}{f}\right)\right] \\ &= 0 \end{aligned}$$

and we move on to the first derivative. Again, it is convenient to consider the values of all $L'(0, \chi, f)$ with $\chi(-1) = 1$ simultaneously. The result is

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$$L'(0,\chi,f) = -\frac{1}{2} \sum_{\substack{a=1 \\ (a,f)=1}}^{f/2} \chi(a) \log \left[(1-e^{2\pi ia/f})(1-e^{-2\pi ia/f})\right] \quad (f>2),$$

$$L'(0,\chi_1,f) = -\frac{1}{2}\log(2) \quad (f=2).$$

This may be rephrased as follows. Let $K = \mathrm{Re}(\mathbb{Q}(e^{2\pi i/f}))$ and $G = \mathrm{Gal}(K/\mathbb{Q}) = (\mathbb{Z}/f\mathbb{Z})^\times/\pm 1$. There is an integer ε in K such that for all characters χ of G,

$$L'(0,\chi,f) = -\frac{1}{2} \sum_{g\in G} \chi(g) \log(\varepsilon\circ g).$$

In this formula, all the $\varepsilon\circ g$ are positive. This is explained by the fact that $K(\sqrt{\varepsilon})$ is abelian over $\mathbb{Q}$ and hence all the $\varepsilon\circ g$ must have the same sign. The same result now holds for any subfield of K by taking relative norms.

Theorem [7]. **If** K **is a totally real abelian extension of** $\mathbb{Q}$ **with Galois group** G **and conductor dividing** $f>1$, **there is a totally positive integer** ε **of** K **such that for all** χ **of** G,

$$L'(0,\chi,f) = -\frac{1}{2} \sum_{g\in G} \chi(g) \log(\varepsilon\circ g).$$

Corollary. If f **is a power of a prime** p, **then** $\mathbb{N}(\varepsilon) = p$. **Otherwise** $\mathbb{N}(\varepsilon) = 1$ **and** ε **is a unit.**

In particular the $\varepsilon\circ g$ are all associates since when f is a power of p, p ramifies totally. The Corollary follows easily from the Theorem by using the principal character along with the observation that

$$\zeta'(0,f) = \begin{cases} -\frac{1}{2}\log(p) & \text{if f is a power of p} \\ 0 & \text{otherwise.} \end{cases}$$

Let us give two examples of the uses of the Theorem. First suppose that $K = \mathbb{Q}(\sqrt{p})$ where p is a prime congruent to 1 (mod 4). We take $f=p$. There are two characters, the trivial character χ_1, and a quadratic character χ_p. We have

$$-\frac{1}{2}\log(p) = L'(0,\chi_1,p) = -\frac{1}{2}\log(ab)$$

$$h_p \log(\varepsilon_p) = L'(0,\chi_p,p) = -\frac{1}{2}\log(a/b)$$

where a and b are the two conjugates of ε, both positive, h_p and ε_p are the class-number and fundamental unit of $\mathbb{Q}(\sqrt{p})$. Hence

$$\varepsilon_p{}^{h_p} = \left(\frac{b}{a}\right)^{1/2} = \frac{\sqrt{p}}{a}$$

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and so

$$\mathbb{N}(\varepsilon_p{}^{h_p}) = \frac{-p}{p} = -1$$

which gives the old result that h_p is odd and ε_p has norm -1.

Next, consider the case of two primes p and q both congruent to 1 (mod 4). We take $f = pq$ and $K = (\sqrt{p},\sqrt{q})$. If we denote by g_p, g_q, g_{pq} the elements of G fixing $\sqrt{p},\sqrt{q}$, $\sqrt{pq}$ respectively, then we have with the obvious notation.

$$0 = L'(0,\chi_1,pq) = -\frac{1}{2}\log(\varepsilon\ \varepsilon\circ g_p\ \varepsilon\circ g_q\ \varepsilon\circ g_{pq}),$$

$$(h_p \log(\varepsilon_p))\,(1-\chi_p(q)) = L'(0,\chi_p,pq) = -\frac{1}{2}\log\left(\frac{\varepsilon\ \varepsilon\circ g_p}{\varepsilon\circ g_q\ \varepsilon\circ g_{pq}}\right),$$

$$(h_q \log(\varepsilon_q))\,(1-\chi_q(p)) = L'(0,\chi_q,pq) = -\frac{1}{2}\log\left(\frac{\varepsilon\ \varepsilon\circ g_q}{\varepsilon\circ g_p\ \varepsilon\circ g_{pq}}\right),$$

$$h_{pq}\log(\varepsilon_{pq}) = L'(0,\chi_{pq},pq) = -\frac{1}{2}\log\left(\frac{\varepsilon\ \varepsilon\circ g_{pq}}{\varepsilon\circ g_p\ \varepsilon\circ g_q}\right).$$

There are two cases to consider. If $\chi_p(q) = \chi_q(p) = 1$, then

$$\varepsilon = \varepsilon\circ g_{pq} = (\varepsilon\circ g_p)^{-1} = (\varepsilon\circ g_q)^{-1} = \varepsilon_{pq}{}^{-h_{pq}/2}.$$

Among other things, this says that $\varepsilon = \varepsilon\circ g_{pq}$ is in $\mathbb{Q}(\sqrt{pq})$ and so h_{pq} is even. Further, $\varepsilon\circ g_p > 0$ and so

$$\mathbb{N}(\varepsilon_{pq}{}^{-h_{pq}/2}) = 1.$$

Therefore if ε_{pq} has norm -1, then 4 divides h_{pq}.

In the other case, we have $\chi_p(q) = \chi_q(p) = -1$ and then we find that

$$\varepsilon_{pq}{}^{h_{pq}/2} = \frac{\varepsilon_p{}^{-h_p}\,\varepsilon_q{}^{-h_q}}{\varepsilon}$$

is in $\mathbb{Q}(\sqrt{p},\sqrt{q})$. From this, we see that

$$(\varepsilon_{pq}{}^{h_{pq}/2})\circ g_{pq} = \frac{(-\varepsilon_p{}^{h_p})\,(-\varepsilon_q{}^{h_q})}{\varepsilon\circ g_{pq}} = \varepsilon_{pq}{}^{h_{pq}/2}$$

and thus $\varepsilon_{pq}{}^{h_{pq}/2}$ is in $\mathbb{Q}(\sqrt{pq})$ and therefore h_{pq} is even. Note that it is only necessary to know the sign of $(\varepsilon_{pq}{}^{h_{pq}/2})\circ g_{pq}$ to tell whether it is in $\mathbb{Q}(\sqrt{pq})$ or generates $\mathbb{Q}(\sqrt{p},\sqrt{q})$ over $\mathbb{Q}(\sqrt{pq})$. In like manner, the conjugate of $\varepsilon_{pq}{}^{h_{pq}/2}$ is

$$(\varepsilon_{pq}{}^{h_{pq}/2})\circ g_p = \frac{(\varepsilon_p{}^{-h_p})\,(-\varepsilon_q{}^{h_q})}{\varepsilon\circ g_p}$$

is negative and so ε_{pq} has norm -1. This result goes back to Dirichlet.

Before we leave the ground field of the rational numbers, let us note one more property of the values at $s=0$. The interpretation of $\zeta_K(s,f)$ as the zeta function of the Dedekind domain $A = \mathfrak{O}_K[1/f]$ suggests that the lead terms of the power series

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expansion for each $L(s,\chi,f)$ at $s=0$ can have an interpretation as a factor of R(A) and leads to the same refined conjectures as for factors of R(K).

For instance, suppose $K=\mathbb{Q}(e^{2\pi i/f})$ and that p is a prime congruent to 1 (mod f) so that p splits completely in K. If we develop a conjecture in the same way as in [6] and [7], we might suspect that there is an element ε of $\mathfrak{O}_K[1/p]$ (i.e., ε is integral outside p) such that $|\varepsilon \circ g| = 1$ for all g in $G=\mathrm{Gal}(K/\mathbb{Q})$, $K(\varepsilon^{1/W(K)})$ is abelian over $\mathbb{Q}$ while

$$L'(0,\chi,fp) = -\frac{1}{W(K)} \sum_{g\in G} \chi(g)\log \|\varepsilon\circ g\|_v,$$

where v is the normalized valuation of K corresponding to a prime ideal in K above p. But we already know that

$$L'(0,\chi,fp) = L(0,\chi,f)\log(p) = \sum_{\substack{a=1\\(a,f)=1}}^{f} \chi(a)\left(\frac{f-2a}{2f}\right)\log(p).$$

Therefore, if we identify g in G with an integer a in the isomorphic group $(\mathbb{Z}/f\mathbb{Z})^\times$ with $0<a<f$, the conjecture leads to

$$\|\varepsilon\circ g\|_v = p^{\frac{W(K)}{2f}(2a-f)}$$

for all g in G. Tate [8] and Gross [3] have both realized that this gives the complete ideal factorization of (ε); the fact that ε is a number is Stickelberger's theorem and the rest of the conjecture follows from the properties of the Gauss sums used to prove Stickelberger's theorem. Even over the rational numbers, new discoveries are being made linking Dedekind zeta functions and Dedekind domains.

3. Values of zeta functions for totally real fields

We now turn to other ground fields. The problem of finding a closed form expression for the values at zero and negative integers of the zeta and L-functions of a field k was first solved by Shintani [4]. Prior to this, Siegel [5] had shown that the partial zeta functions took rational values (without finding them) and Zagier [9] found them for real quadratic fields at all negative integers and at zero. We illustrate the situation for real quadratic fields but the methods are general. For real quadratic fields, the problem ultimately becomes one of evaluating the function

$$\zeta(s_1,s_2,x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (am+bn+x)^{-s_1}(cm+dn+y)^{-s_2},$$

where a, b, c, d are positive real numbers, $ad-bc>0$. The series converges for $\mathrm{Re}(s_1+s_2)>2$ and positive x and y. Shintani's loop integral expression for $\zeta(s_1,s_2,x,y)$ provides the analytic continuation necessary to justify the following.

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In analogy with the second section, we have

$$(8)\qquad \frac{\partial}{\partial x}\,\zeta(s_1,s_2,x,y) = -s_1\,\zeta(s_1+1,s_2,x,y),$$

$$(9)\qquad \frac{\partial}{\partial y}\,\zeta(s_1,s_2,x,y) = -s_2\,\zeta(s_1,s_2+1,x,y).$$

Further, for $\mathrm{Re}(s_1+s_2) < 2$, we have the formula,

$$(10)\qquad \int_0^1\int_0^1 \zeta(s_1,s_2,ax+by,cx+dy)\,dx\,dy = 0,$$

although the proof is more difficult: we have

$$a\int_0^1\int_0^1 (s_1-1)\,\zeta(s_1,s_2,ax+by,cx+dy)\,dx\,dy +$$

$$c\int_0^1\int_0^1 s_2\,\zeta(s_1-1,s_2+1,ax+by,cx+dy)\,dx\,dy$$

$$= \int_0^1 -\zeta(s_1-1,s_2,ax+by,cx+dy)\,\Big|_{x=0}^{x=1}\,dy$$

$$= \int_0^1 \sum_{n=0}^{\infty} (bn+by)^{s_1-1}\,(dn+dy)^{s_2}\,dy$$

$$= 0.$$

Likewise,

$$b\int_0^1\int_0^1 (s_1-1)\,\zeta(s_1,s_2,ax+by,cx+dy)\,dy\,dx +$$

$$d\int_0^1\int_0^1 s_2\,\zeta(s_1-1,s_2+1,ax+by,cx+dy)\,dy\,dx$$

$$= 0$$

from which (10) follows.

For example, by comparison with the integral, it is easy to see that

$$s\,(s+1)\,\zeta(s+2,s,x,y)|_{s=0} = \mathrm{res}_{s=2}\sum_{m=0}\ \sum_{n=0}(am+bn+x)^{-s} = \frac{1}{ab}.$$

Hence,

$$s\,\zeta(s+1,s,x,y)|_{s=0} = -\,\frac{x}{ab} + g(y)$$

for some function $g(y)$. But further,

$$s^2\,\zeta(s+1,s+1,x,y)|_{s=0} = 0,$$

and so,

$$s\,\zeta(s+1,s,x,y)|_{s=0} = -\,\frac{x}{ab} + C$$

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where C is constant. But by (10),

$$C=\int_0^1\int_0^1 \frac{ax+by}{ab}\,dx\,dy=\frac{a+b}{2ab},$$

$$s\,\zeta(s+1,s,x,y)|_{s=0}=-\frac{x}{ab}+\frac{a+b}{2ab}.$$

By interchanging variable names, we see that

$$s\,\zeta(s,s+1,x,y)|_{s=0}=-\frac{y}{cd}+\frac{c+d}{2cd}.$$

Hence,

$$\zeta(0,0,x,y)=\frac{1}{2ab}\,[x^2-(a+b)x]+\frac{1}{2cd}\,[y^2-(c+d)y]+C$$

for some constant C. Again, we find C from (10), which gives

$$\zeta(0,0,x,y)=\frac{1}{12ab}\,[6x^2-6x(a+b)+(a^2+3ab+b^2)]$$
$$+\frac{1}{12cd}\,[6y^2-6y(c+d)+(c^2+3cd+d^2)].$$

This is the formula found by Shintani and Zagier.

4. First derivatives at $s=0$

We now turn to first derivatives at zero of L-functions defined over number fields. As an example, let us take the case of a pure cubic field, say $K=\mathbf{Q}(\sqrt[3]{2})$. Dedekind showed using the cubic reciprocity law that

$$\zeta_K(s)=\zeta(s)\,L(s,\chi)$$

where $L(s,\chi)$ is an abelian L-function defined over $k=\mathbf{Q}(\sqrt{-3})$. The analytic continuation and functional equation of $\zeta_K(s)$ follow. This was one of the main examples pointing the way towards Hecke's eventual general result. This example also anticipated Artin's general theory of non-abelian L-functions and their connection with abelian L-functions via the reciprocity law. In this regard, it should also be mentioned that the requisite theory of representations of non-abelian finite groups was inspired by the letter exchange between Dedekind and Frobenius.

For our purposes, we see that

$$L'(0,\chi)=h(K)\,\log(\varepsilon_K)$$

where $h(K)=1$ is the class-number of K and $\varepsilon_K=(\sqrt[3]{2}-1)^{-1}$ is the fundamental unit of K. This illustrates a special case of the only other type ground field where my first order zero conjectures are proved. The situation is as follows. Suppose that k is a complex quadratic field and K is a class field of k whose conductor divides an ideal $\mathfrak{f}\neq(1)$. We let $G=\mathrm{Gal}(K/k)$, $W=W(K)$ and also we use $\|\alpha\|=|\alpha|^2$ to denote the normalized archemedian valuation of K.

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Theorem [7]. **There is an integer** ε **in** K **such that** $\varepsilon \circ g$ **is an associate of** ε **for all** g **in** G, $K(\varepsilon^{1/W})$ **is an abelian extension of** k **and for every** χ **of** G,

$$L'(0,\chi,\mathfrak{f}) = -\frac{1}{W}\sum_{g\in G}\chi(g)\log\|\varepsilon\circ g\|.$$

If $\mathfrak{f}$ **has two distinct prime ideal factors,** ε **is a unit.**

The proof of this theorem makes use of Kronecker's second limit formula. Let $\Omega=[\omega_1,\omega_2]$ denote the lattice generated by ω_1 and ω_2 ordered such that $z=\omega_2\omega_1^{-1}$ is in the upper half plane. For w not in Ω, Kronecker's second limit formula at $s=0$ is

$$\left[\Gamma(s)\sum_{\omega\in\Omega}|\omega+w|^{-2s}\right]\Big|_{s=0} = -\log\left[\left|F\left(\frac{w}{\omega_1},z\right)\right|^2\right],$$

where

$$F(\gamma,z)=\exp\left[\pi i\left(\frac{\gamma-\bar\gamma}{z-\bar z}\right)\right]\frac{\vartheta_1(\gamma,z)}{\eta(z)},$$

$\vartheta_1(\gamma,z)$ is Jacobi's theta function and $\eta(z)$ is the Dedekind eta function. When γ is of the form $\frac{az+b}{N}$ with a and b in $\mathbb{Z}$, $F(\gamma,z)$ is a modular function of level $12N^2$. When z is in k, this puts $F(\gamma,z)$ in a class field of k. However, the fact that $F(\gamma,z)$ is nearly a unit is independent of this. So long as $j(z)$ is an algebraic integer, $F(\gamma,z)$ will be an N-unit (in other words, a unit at all primes not dividing N) and indeed will be a unit if N has at least two distinct prime factors. If $j(z)$ is algebraic with denominator dividing Δ, then $F(\gamma,z)$ will be an $N\Delta$-unit and in fact will be just a Δ-unit if N has at least two distinct prime factors. This opens up some interesting non-abelian possibilities and I would like to close this paper by briefly discussing them.

Recall that any elliptic function may be expressed as a theta quotient. As a result, any elliptic function may be expressed via Kronecker's limit formula. For instance, suppose that E is an elliptic curve in Weierstrass normal form,

$$y^2=4x^3-g_2x-g_3,$$

with discriminant $\Delta(E)=g_2^3-27g_3^2$. If Ω is the corresponding lattice, we have $\Delta(E)=(2\pi/\omega_1)^{12}\eta(z)^{24}$. Comparing zeros and poles shows that

$$\frac{\wp(w)-\wp(w_0)}{\Delta(E)^{1/6}}=\frac{F\left(\frac{w+w_0}{\omega_1},z\right)\,F\left(\frac{w-w_0}{\omega_1},z\right)}{F\left(\frac{w_0}{\omega_1},z\right)^2\,F\left(\frac{w}{\omega_1},z\right)^2},$$

for an appropriate sixth root of $\Delta(E)$ and hence,

$$\left\{\Gamma(s)\sum_{\omega\in\Omega}\left[|\omega+w+w_0|^{-2s}+|\omega+w-w_0|^{-2s}-2|\omega+w|^{-2s}-2|\omega+w_0|^{-2s}\right]\right\}\Big|_{s=0} = -\log\left[\left|\frac{\wp(w)-\wp(w_0)}{\Delta(E)^{1/6}}\right|^2\right].$$

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A similar result holds for $\wp'(w)$,

$$\left\{\Gamma(s) \sum_{\omega\in\Omega} [|\omega+2w|^{-2s} - 4|\omega+w|^{-2s}]\right\}\Bigg|_{s=0} = -\log\left[\left|\frac{\wp'(w)}{\Delta(E)^{1/4}}\right|^2\right].$$

In particular, if g_2 and g_3 are integral, and w and w_0 are N-division points with neither w nor w_0 nor $w+w_0$ nor $w-w_0$ in Ω, or simply 2w is not in Ω in the second case, then

$$\frac{\wp'(w)-\wp'(w_0)}{\Delta(E)^{1/6}} \text{ and } \frac{\wp'(w)}{\Delta(E)^{1/4}}$$

are $N\Delta$-units with the indicated improvements holding for N having at least two distinct prime factors or j(z) being integral. These numbers even arise from the first derivative at $s=0$ of generalized Dirichlet series as the formulas above show. Whether or not they also arise naturally as units in my conjectures is not known, but we have, at the very least, an extremely interesting supply of new units to think about.

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